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Illustrative examples of anisotropic friction with sliding path curvature effects

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Abstract

A constitutive equation of anisotropic friction with sliding path curvature effects defined in the preceding companion paper is completed with illustrative examples. Friction coefficients, inclination angles and coefficients of tangent and normal components of the friction force with respect to the sliding direction are given in the case of non-homogeneous friction properties which form concentric circles in a contact surface. Motion of a material point in the surface with non-homogeneous friction is investigated for radial, concentric circular and arbitrary trajectories. Essential changes of sliding trajectories of the material point are observed for various values of coefficients of parametric tensors in the constitutive equation. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

A knowledge of a frictional resistance and a wear behaviour of materials with complex microstructure has a great importance with regard to their applications as machinery component parts subject to contact and sliding motion. Anisotropy, non-homogeneity and sliding path curvature effects are clearly specified tribological properties of these materials. In the large body of literature devoted to friction and wear, anisotropy was observed by many researchers, the sliding path curvature effects were treated by Briscoe and Stolarski (1979, 1985, 1986, 1991). The present analysis is supported by experimental observations carried out by Briscoe and Stolarski.

We discuss the sliding status in the case when the sliding path curvature modifies the expression of the friction force. The papers by Zmitrowicz (1995, 1999) formulated the friction force constitutive equation with sliding path curvature effects in the following form

$$\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_\rho = -N\left(\mathbf{C}\mathbf{v} + \mathbf{E}\frac{\mathbf{n}}{\rho}\right), \tag{1}$$

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where, $N \geqslant 0$ is a normal pressure, ρ is a curvature radius, $\mathbf{v}$ and $\mathbf{n}$ are unit vectors tangent and normal to the sliding path, respectively.

Let us consider non-homogeneous friction properties which form concentric circles in the surface with a centre in the origin of the reference system Oxy. Then, the second order tensor $\mathbf{C}$ is defined as follows

$$\mathbf{C} = \mu_1 \mathbf{k}_1 \otimes \mathbf{k}_1 + \mu_2 \mathbf{k}_2 \otimes \mathbf{k}_2. \tag{2}$$

Unit vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are tangent and normal to the concentric circles, and coefficients μ_1 and μ_2 define friction along the concentric circles and along the radii, respectively. Tensor $\mathbf{E}$ is associated with the sliding motion and is given by

$$\mathbf{E} = \eta_1 \mathbf{v} \otimes \mathbf{n} + \eta_2 \mathbf{n} \otimes \mathbf{n}, \tag{3}$$

where η_1 and η_2 are coefficients. The basis $\{\mathbf{k}_1, \mathbf{k}_2\}$ can be transformed into the basis $\{\mathbf{v}, \mathbf{n}\}$ with the aid of the following rule

$$[\mathbf{k}_1, \mathbf{k}_2]^T = \mathbf{B}[\mathbf{v}, \mathbf{n}]^T, \tag{4}$$

where, $\mathbf{B} = [B_{ij}]$, $i, j = 1, 2$ is a transformation matrix.

If $\rho > 0$, then two types of restrictions can be imposed on the tensors $\mathbf{C}$ and $\mathbf{E}$. Let the tensors satisfy the following conditions

$$\mathbf{v}^T\mathbf{C}\mathbf{v} \geqslant 0, \quad \mathbf{v}^T\mathbf{E}\mathbf{n} \geqslant 0, \tag{5}$$

then the dissipation inequality (Zmitrowicz, 1999) is satisfied for every sliding direction and every positive radius of curvature $\rho \in R^+$. If $\rho > 0$ and the tensors are restricted as follows

$$\mathbf{v}^T\mathbf{C}\mathbf{v} > 0, \quad \mathbf{v}^T\mathbf{E}\mathbf{n} \leqslant 0, \tag{6}$$

then the dissipation inequality (Zmitrowicz, 1999) is satisfied for every sliding direction and some values of the positive curvature radii, i.e.

$$\rho \geqslant -\frac{\mathbf{v}^T\mathbf{E}\mathbf{n}}{\mathbf{v}^T\mathbf{C}\mathbf{v}}. \tag{7}$$

In illustrative examples, we apply restrictions (6) and (7) since they lead to numerical results which coincide with experimental observations carried out by Briscoe and Stolarski (1979, 1985, 1986, 1991).

The anisotropic friction coefficient μ_α and the angle of friction force inclination β to the sliding direction are defined by

$$\mu_\alpha = N^{-1}|\mathbf{t}|, \tag{8}$$

$$\sin\beta = \frac{\mathbf{t}\cdot\mathbf{n}}{|\mathbf{t}|}. \tag{9}$$

Coefficients of the friction force components collinear with the sliding direction and normal to the sliding direction are as follows

$$\mu_\alpha^{\|} = -N^{-1}\mathbf{t}\cdot\mathbf{v}, \quad \mu^{\perp}{}_\alpha = N^{-1}\mathbf{t}\cdot\mathbf{n}. \tag{10}$$

2. Characteristic friction quantities

Characteristic quantities of the analysed anisotropic non-homogeneous friction (i.e. μ_α, β, $\mu_\alpha^{\|}$, $\mu^{\perp}{}_\alpha$) depend not only on the sliding direction in the contact but also on the shape of the sliding path. We consider two privileged sliding trajectories i.e. along the radii and along the concentric circles. Characteristic friction quantities in these cases are represented with the aid of simple formulae.

(a) Radial trajectories. Let us assume that the sliding trajectories are straight lines collinear with the radii from the centre of concentric circles. Since the following relations hold between the unit vectors: $\mathbf{k}_1 = \mathbf{n}$, $\mathbf{k}_2 = -\mathbf{v}$, the friction is defined with the aid of the tensor $\mathbf{C}$ given by

$$\mathbf{C} = \mu_1\mathbf{n}\otimes\mathbf{n}+\mu_2\mathbf{v}\otimes\mathbf{v}. \tag{11}$$

For arbitrary tensor $\mathbf{E}$, we have $\mathbf{t}_\rho = \mathbf{0}$, since $\rho = \infty$. Then, the friction force equation reduces to the following form

$$\mathbf{t}\equiv\mathbf{t}_0 = -\mu_2 N\mathbf{v}. \tag{12}$$

In that case, the coefficients of the friction force components and the inclination angle are defined by

$$\mu_\alpha \equiv \mu_\alpha^{\|} = \mu_2, \quad \mu^{\perp}{}_\alpha, \beta = 0. \tag{13}$$

Here, the friction force vector is always collinear with the radial direction, and the friction force value is constant for all radii. A power of the friction force referred to the unit velocity (a dissipation function) takes the following value

$$D = -\mathbf{t}\cdot\mathbf{v} = \mu_2 N. \tag{14}$$

After substitution (11) into (6) we obtain a restriction on the friction coefficient μ_2 of the tensor (11) for the sliding along the radii, i.e.

$$\mathbf{v}^T(\mu_1\mathbf{n}\otimes\mathbf{n}+\mu_2\mathbf{v}\otimes\mathbf{v})\mathbf{v} = \mu_2 > 0. \tag{15}$$

The inequality (15) deals with radial sliding directions.

(b) Circular trajectories. If the sliding trajectory is a circle with the radius $\rho = r$ attached to the centre of concentric circles, then $\mathbf{k}_1 = \mathbf{v}$, $\mathbf{k}_2 = \mathbf{n}$, and the friction properties are defined by the tensor $\mathbf{C}$ of the form

$$\mathbf{C} = \mu_1\mathbf{v}\otimes\mathbf{v}+\mu_2\mathbf{n}\otimes\mathbf{n}, \tag{16}$$

and the tensor $\mathbf{E}$ given by (3). Then, the friction force components are defined by

$$\mathbf{t}_0 = -\mu_1 N\mathbf{v}, \tag{17}$$

$$\mathbf{t}_\rho = -\frac{N}{r}(\eta_1 \mathbf{v} + \eta_2 \mathbf{n}), \tag{18}$$

and the friction force equation has the following form

$$\mathbf{t} = -N\left[\left(\mu_1 + \frac{\eta_1}{r}\right)\mathbf{v} + \frac{\eta_2}{r}\mathbf{n}\right]. \tag{19}$$

In this case, the friction coefficient is a function of the coefficients, μ_1, η_1, η_2 and the circle radius r

$$\mu_\alpha = \sqrt{\left(\mu_1 + \frac{\eta_1}{r}\right)^2 + \left(\frac{\eta_2}{r}\right)^2}. \tag{20}$$

Depending on a sign of the coefficient η_1, we distinguish a positive additional friction ($\eta_1 > 0$) and a negative additional friction ($\eta_1 < 0$). The coefficients of the friction force components collinear with the sliding direction and normal to it are given by

$$\mu_\alpha^{\parallel} = \mu_1 + \frac{\eta_1}{r}, \quad \mu^{\perp}{}_\alpha = -\frac{\eta_2}{r}. \tag{21}$$

The angle of the friction force inclination to the sliding direction is defined by

$$\sin\beta = -\frac{\dfrac{\eta_2}{r}}{\sqrt{\left(\mu_1 + \dfrac{\eta_1}{r}\right)^2 + \left(\dfrac{\eta_2}{r}\right)^2}}. \tag{22}$$

The dissipation function takes the following value

$$D = -\mathbf{t}\cdot\mathbf{v} = \left(\mu_1 + \frac{\eta_1}{r}\right)N. \tag{23}$$

After substitution (16) and (3) into (6) we obtain restrictions on the coefficients μ_1 and η_1 of the tensors (16) and (3) for the sliding along the concentric circles, i.e.

$$\mathbf{v}^T(\mu_1 \mathbf{v}\otimes\mathbf{v} + \mu_2 \mathbf{n}\otimes\mathbf{n})\mathbf{v} = \mu_1 > 0, \tag{24}$$

$$\mathbf{v}^T(\eta_1 \mathbf{v}\otimes\mathbf{n} + \eta_2 \mathbf{n}\otimes\mathbf{n})\mathbf{n} = \eta_1 \leqslant 0, \tag{25}$$

since $\mathbf{n}\cdot\mathbf{v} = 0$, $\mathbf{n}\perp\mathbf{v}$. In this case, there are no restrictions on the coefficient η_2, i.e. $\eta_2 \in R$. The investigations (24) and (25) deal with concentric circular trajectories, which satisfy the condition (7), i.e. radii of the circles are greater than a limiting radius (r_l)

$$r \geqslant r_l = -\frac{\eta_1}{\mu_1}. \tag{26}$$

If μ_1, $\mu_2 \neq 0$ and η_1, $\eta_2 = 0$ for the sliding along the concentric circles, then the friction characteristic quantities take the following values: $\mu_\alpha^{\perp}$, $\beta = 0$ and $\mu_\alpha = \mu_\alpha^{\parallel} = \mu_1$. The friction vector is collinear with a tangent to the circle and the friction force value is constant for all concentric circles. Notice, this case takes place, if the tensor **E** is equal to zero, and the friction is defined by the tensor **C**.

Considering the sliding along the concentric circles and taking μ_1, μ_2, $\eta_1 \neq 0$ and $\eta_2 = 0$, the friction force vector is tangent to the circles, but the friction force value depends on the circle radius. In an infinity ($r = \infty$), the friction coefficient is equal to $\mu_\alpha = \mu_1$.

If η_2 is constant during the sliding along the concentric circle of the given radius r, then the friction force vector (19) changes its orientation with respect to the reference system, but its length and the inclination angle are unchanged. Hence, the friction coefficient μ_α and the inclination angle β are constant for all points of the concentric circular trajectory. The inclination angle depends on the circle radius (since $\eta_1 \neq 0$), and the friction force component $\mu_\alpha^\perp$ occurs. For $r = \infty$ the friction coefficient achieves the value $\mu_\alpha = \mu_1$, and $\mu_\alpha^\perp$ and β are equal to zero.

If the coefficient $\eta_2 \neq$ const (e.g. η_2 is a function of the sliding velocity), then the friction depends on: the sliding trajectory and additionally on the sliding velocity at the given point of the trajectory. Therefore, characteristic friction quantities ($\mu_\alpha^\perp, \mu_\alpha, \beta, \mathbf{t}$) are functions of the sliding velocity. Only the coefficient $\mu_\alpha^\parallel$ and the dissipation function D do not depend on the sliding velocity.

Friction process is always connected with wear. Therefore, a relation exists between a wear rate and the friction coefficient. In pin-on-disc tests carried out by Briscoe and Stolarski (1979, 1985, 1991), maximum wear rate of polymers was observed for large radius of the circular path and significant reduction in wear rate was achieved when the radius approached the radius of the pin. The anisotropic friction presented in this study has similar properties. The friction coefficient μ_α for the large radius of the circular trajectory is greater than the friction coefficient for the small radius, taking the negative additional friction ($\eta_1 < 0$).

3. Sliding of a material point

Let us consider motion of a material point in a supporting plane with non-homogeneous anisotropic friction. The motion is described by the following equation

$$m\ddot{\mathbf{x}} = \mathbf{t}, \tag{27}$$

where, m is the mass of the material point, $\mathbf{x}$ is the position vector. The sliding is excited by the initial velocity $\mathbf{V}_0$.

As opposed to the velocity vector, the acceleration vector is not tangent to the material point trajectory in general. The acceleration vector is given by

$$\ddot{\mathbf{x}} = \frac{\mathrm{d}(V\mathbf{v})}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t}\mathbf{v} + \frac{V^2}{\rho}\mathbf{n}, \tag{28}$$

since the unit vector $\mathbf{v}$ is a function of time, and its time derivative has the following form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = V\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}, \tag{29}$$

where, V is a velocity value. The relation (28) defines components of the acceleration tangent and normal to the trajectory (i.e. tangential and centrifugal accelerations). Components of the friction force tangent and normal to the sliding path are given by

$$\mathbf{t} = t^{\parallel}\mathbf{v} + t^{\perp}\mathbf{n}, \tag{30}$$

$$t^{\parallel} = -N\left[\mu_1(B_{11})^2 + \mu_2(B_{21})^2 + \frac{\eta_1}{\rho}\right], \tag{31}$$

$$t^{\perp} = -N\left[\mu_1 B_{11}B_{12} + \mu_2 B_{21}B_{22} + \frac{\eta_2}{\rho}\right]. \tag{32}$$

The motion equation given in the local basis defined by the unit vectors tangent and normal to the sliding trajectory $\{\mathbf{v}, \mathbf{n}\}$ has the following form

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = t^{\parallel},$$

$$m\frac{V^2}{\rho} = t^{\perp}. \tag{33}$$

The following relation holds for the motion along the curve

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{V}{\rho}, \tag{34}$$

where, α is an angle between the axis Ox of the reference system and a tangent to the curve. After substitution (34) into the motion eqn (33) we obtain

$$m\frac{\mathrm{d}V}{\mathrm{d}t} = t^{\parallel},$$

$$mV\frac{\mathrm{d}\alpha}{\mathrm{d}t} = t^{\perp}. \tag{35}$$

Initial conditions of motion are as follows

$$V(t_0) = V_0, \quad \alpha(t_0) = \alpha_0. \tag{36}$$

An instant position of the material point with respect to the reference system Oxy is given by

$$x = x_0 + \int_{t_0}^{t} V\cos\alpha\,\mathrm{d}\tau, \tag{37}$$

$$y = y_0 + \int_{t_0}^{t} V\sin\alpha\,\mathrm{d}\tau, \tag{38}$$

where

$$x(t_0) = x_0, \quad y(t_0) = y_0. \tag{39}$$

The unit vectors associated with the motion trajectory are defined by

$$[\mathbf{v}] = [\cos\alpha, \sin\alpha]^T, \tag{40}$$

$$[\mathbf{n}] = [-\sin\alpha, \cos\alpha]^T. \tag{41}$$

The basis $\{\mathbf{k}_1, \mathbf{k}_2\}$ of the concentric circles is given by

$$[\mathbf{k}_1] = \frac{1}{\sqrt{x^2+y^2}}[y, -x]^T, \tag{42}$$

$$[\mathbf{k}_2] = \frac{1}{\sqrt{x^2+y^2}}[x, y]^T. \tag{43}$$

The normal pressure force N is taken to be equal to the gravity force Newton $= 9.81N$ and $\mu_1 = 0.1$, $\mu_2 = 0.15$, $\eta_1 = -0.01m$, in all examples in this study.

A resistance to sliding of the material point depends on frictional properties of the supporting surface. In the case of non-homogeneous anisotropic friction, the resistance to sliding depends on the sliding trajectory.

(a) Radial trajectories. Let us consider the material point sliding in the radial direction. The sliding is excited by the initial velocity $\mathbf{V}_0$ acting at the point (x_0, y_0) in the radial direction α_0. Then, coefficients of the transformation matrix $\mathbf{B}$ have the following values

$$B_{11} = B_{22} = 0, \quad B_{12} = -B_{21} = 1, \tag{44}$$

and the friction force components are as follows

$$t^{\parallel} = -\mu_2 N, \quad t^{\perp} = 0. \tag{45}$$

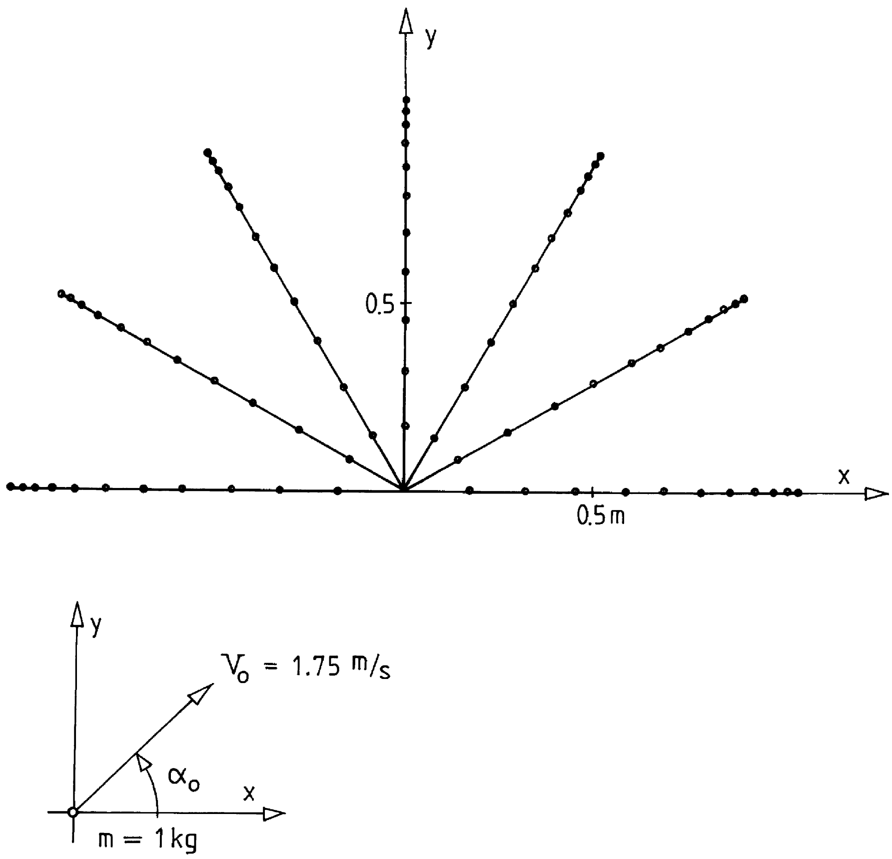
Then the motion eqns (35) reduce to

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{N}{m}\mu_2, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}t} = 0. \tag{46}$$

The material point which slides due to the initial velocity acting in the radial direction does not leave the radial direction, since $\beta = 0$ and there is no friction force component being able to change the rectilinear trajectory of the material point. The friction force and the sliding velocity are collinear vectors, and they guarantee the motion along the radii. The coefficients η_1 and η_2 do not affect the sliding along the radii, since the sliding path is a straight line $(1/\rho = 0)$.

In this case the material point motion is a uniformly retarded motion along the radii. The tangential acceleration is constant, i.e. $\ddot{x}_t = -N\mu_2/(2m)$. The centrifugal acceleration is equal to zero. The sliding way passed by the material point is given by

$$s(t) = V_0 t - \frac{N}{2m}\mu_2 t^2. \tag{47}$$



x_0	y_0	α_0						
0	0	0°	30°	60°	90°	120°	150°	180°

Fig. 1. Motion of a material point along the radial trajectories in a plane with friction properties which form concentric circles (starting from the centre of the concentric circles).

Hence, the length of the rectilinear sliding trajectory and the time of sliding are the same for all radial directions, i.e. for any direction angle $\alpha_0 \langle 0, 2\pi \rangle$, Fig. 1.

The motion along the radii does not depend on the initial position (x_0, y_0) of the material point with respect to the concentric circle centre, and it does not depend on the sense of the initial velocity vector $\mathbf{V}_0$ (i.e. into the centre or out of the centre).

Intervals between points on the trajectories plotted in this study correspond to constant time intervals (0.1 s).

(b) Circular trajectories. Let us consider the sliding of the material point along the concentric circle excited by the initial velocity $\mathbf{V}_0$ acting tangent to the circle at the point (x_0, y_0). In this case, we have

$$B_{11} = B_{22} = 1, \quad B_{12} = B_{21} = 0, \tag{48}$$

$$t^{\parallel} = -\left(\mu_1 + \frac{\eta_1}{r}\right)N, \quad t^{\perp} = -\frac{\eta_2}{r}N. \tag{49}$$

The circle radius is given by

$$r = \sqrt{(x_0)^2 + (y_0)^2}. \tag{50}$$

The constraints normal to the circular trajectory are defined with the aid of the coefficient η_2 which is assumed to be the following velocity function

$$\eta_2 = -\frac{mV^2}{N}. \tag{51}$$

A value of the coefficient η_2 is negative, since $m > 0$, $N > 0$ and $V^2 > 0$.

After substitution (51) into (49), the gyroscopic component of the friction force takes the following form

$$t^{\perp} = m\frac{V^2}{r} \tag{52}$$

and the second equation of motion (33) is satisfied identically. Taking into account (35), the sliding along the concentric circle is described by the following equations

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{N}{m}\left(\mu_1 + \frac{\eta_1}{r}\right),$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{V}{r}. \tag{53}$$

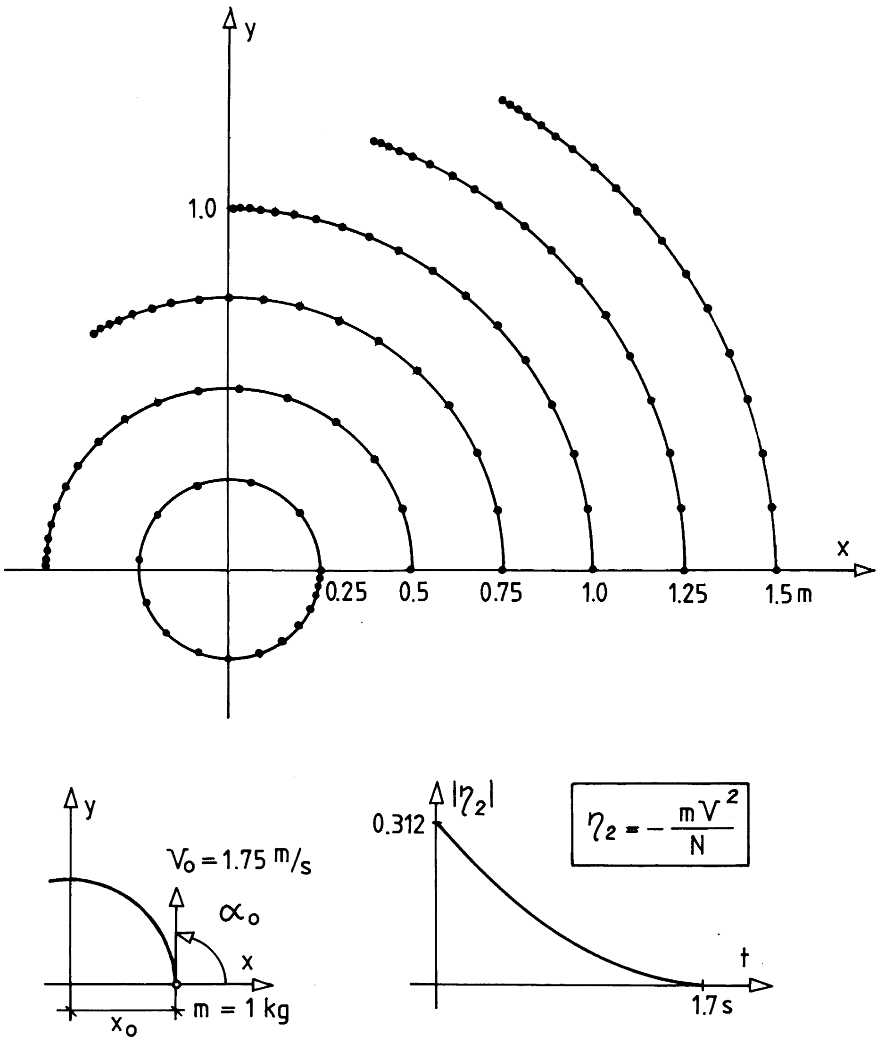
In this case the material point motion is a uniformly retarded motion along the circle of the radius r. The tangential acceleration is constant, i.e. $\ddot{x}_t = -N(\mu_1 + \eta_1/r)/(2m)$.

The motion in direction normal to the circular trajectory is constrained. The material point is constrained only to move along the concentric circular trajectory (an effect of a 'motion by rails'). The reaction to the constraints ($t^{\perp}$) equilibrates the centrifugal force. This example is equivalent to the case when the material point of the mass m is attached to a massless rigid arm free-fixed in the centre of a circle. Then, the material point can move along the circle of the radius equal to the arm length, and the component $t^{\perp}$ defines a force in the arm ($t^{\perp}$ is equal to the centripetal force).

The sliding way passed by the material point is given by

$$s(t) = r\alpha(t) = V_0 t - \frac{N}{2m}\left(\mu_1 + \frac{\eta_1}{r}\right)t^2. \tag{54}$$

Let us assume that $\eta_1 = 0$, then the length of the circular sliding path and the time of sliding are



x_0						y_0	α_0
0.25	0.5	0.75	1.0	1.25	1.5	0	90°

Fig. 2. Motion of a material point along the concentric circular trajectories in a plane with friction properties which form concentric circles (starting from various points in Ox-axis), when $\eta_1 = 0$, $\eta_2 \neq$ const.

constant for all concentric circular trajectories, i.e. for arbitrary and finite radius r, Fig. 2. Since the additional friction is equal to zero ($\eta_1 = 0$), the concentric circular trajectory is the trajectory of the minimum of dissipated energy. In this case, the sliding path is the longest.

The velocity function $V(t)$ does not depend on the initial position (x_0, y_0) and on the direction angle (α_0). Thus, the coefficient η_2 is a function of time having the same values for all concentric circles.

(c) Other trajectories. We consider the material point sliding in arbitrary directions taking into account a few particular values of the coefficients of the tensors **C** and **E**. The friction force components $t^{\parallel}$ and $t^{\perp}$ are defined by (31) and (32). The motion equations (33) we solved by means of the Runge–Kutta fourth-order method.

Isotropic friction ($\mu_1 = \mu_2 \neq 0$ and η_1, $\eta_2 = 0$) gives the rectilinear sliding trajectory which coincides with the direction of the initial velocity $\mathbf{V}_0$. In the case of isotropic friction all sliding directions are equivalent, they give the same resistance to motion. Motion of the point is a uniformly retarded motion.

Figure 3 presents the sliding trajectories for very particular friction properties. In all cases the tensor **E** is equal to zero (i.e. η_1, $\eta_2 = 0$). In the frictionless case (μ_1, μ_2, η_1, $\eta_2 = 0$) the material

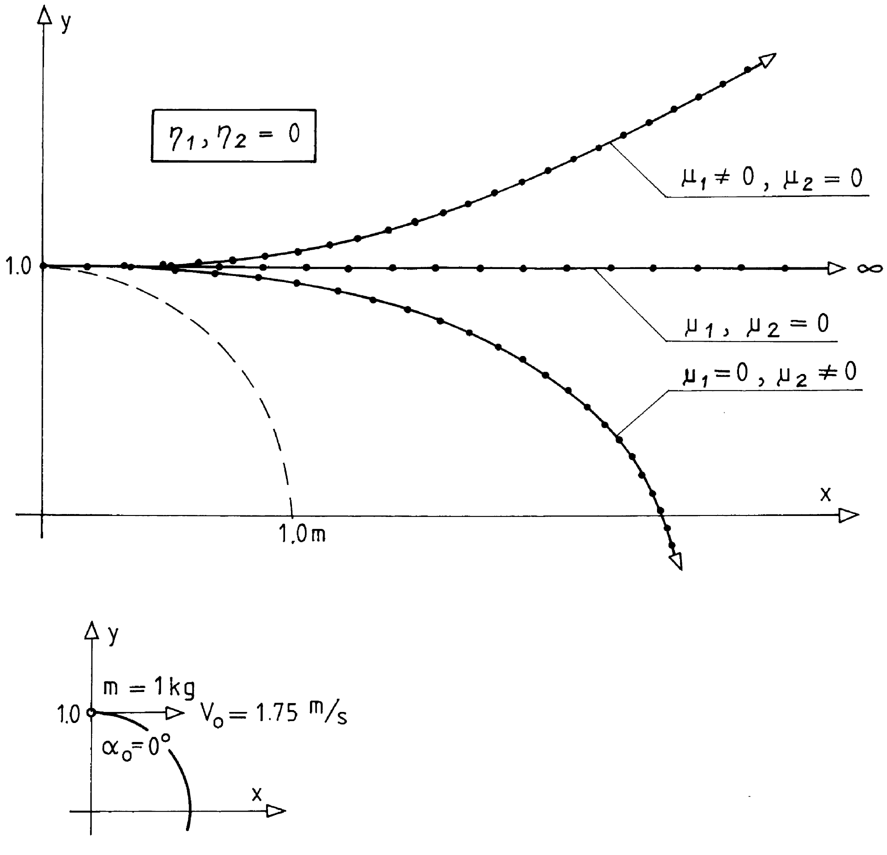


Fig. 3. Motion of a material point in a plane with: frictionless properties (μ_1, $\mu_2 = 0$), unidirectional friction with non-zero friction for the sliding along the concentric circles ($\mu_1 \neq 0, \mu_2 = 0$), unidirectional friction with non-zero friction for the sliding along the radii ($\mu_1 = 0, \mu_2 \neq 0$).

point moves uniformly with the constant velocity V_0 into an infinity along the rectilinear trajectory which coincides with the initial velocity direction. We obtain the same situation in the case μ_1, $\mu_2 = 0$ and $\eta_1, \eta_2 \neq 0$, since the effects of the coefficients η_1 and η_2 vanish for the rectilinear sliding trajectories. It means that the tensor **E** is an addition to the friction description with the aid of the tensor **C**. The tensor **E** does not play any independent role.

Unidirectional friction with non-zero friction for concentric circular trajectories ($\mu_1 \neq 0$ and μ_2, η_1, $\eta_2 = 0$) gives the material point path which bends to the frictionless radial direction, Fig. 3. If the non-zero friction is along the radii ($\mu_2 \neq 0$ and μ_1, η_1, $\eta_2 = 0$), then the sliding path bends to the frictionless concentric circular direction, Fig. 3. Generally, the unidirectional friction gives the trajectory which bends to the frictionless direction (i.e. to the direction without the resistance to motion). Then, the dissipation of the initial kinetic energy of the material point is small.

If the radial friction is greater than the concentric circular friction ($\mu_1 < \mu_2 \neq 0$ and η_1, $\eta_2 = 0$) then the trajectory curve bends to the concentric circles i.e. to the direction with a lower friction. If the concentric circular friction is greater than the radial friction ($\mu_1 > \mu_2 \neq 0$ and η_1, $\eta_2 = 0$), then the motion curve bends to the radial direction. Generally, the motion trajectory bends to the direction with a lower friction.

If $\mu_1, \mu_2, \eta_2 \neq 0$ and $\eta_1 = 0$, then the gyroscopic component of the vector $\mathbf{t}_\rho$ is equal to $-(N\eta_2/\rho)\mathbf{n}$. Since $\eta_2 \neq 0$, the material point moves along the essentially different trajectory curve in comparison with the case: μ_1, $\mu_2 \neq 0$ and η_1, $\eta_2 = 0$. The sliding velocity, the time of sliding and the resistance to motion do not change radically, and they are almost the same in both cases. Very small changes arise from different values of the transformation coefficients B_{ij} for different sliding curves.

Figure 4 illustrates an influence of the coefficient μ_2 on the sliding trajectory. Two particular types of the coefficient η_2 are taken into account: $\eta_2 = \text{const}$ and $\eta_2 \neq \text{const}$. If $\eta_2 = \text{const}$ and it is positive, then the sliding path is inclined with respect to the straight line collinear with the initial velocity $\mathbf{V}_0$. If $\eta_2 \neq \text{const}$ and it is assumed to be the following velocity function

$$\eta_2 = -\eta_2^* \frac{mV^2}{N}, \tag{55}$$

where $0 < \eta_2^* < 1$, then the sliding path bends essentially to the concentric circle, and it coincides with the concentric circle in the case $\eta_2^* = 1$.

Figure 5 shows an influence of the coefficient $\eta_1 < 0$ on the material point motion. The coefficient $\eta_1 = -0.01m$ defines the negative additional friction. Then the dissipative component of the vector $\mathbf{t}_\rho$ is equal to $-(N\eta_1/\rho)\mathbf{v}$. The negative additional resistance to motion is great for trajectories with small curvature radius ρ, and opposite the negative additional friction is small for trajectories with great ρ. If $\eta_1 < 0$, then the material point experiences the smaller resistance to motion in comparison with the case: μ_1, μ_2, $\eta_2 \neq 0$ and $\eta_1 = 0$. The sliding velocity and the time of sliding change essentially, but the trajectory curve does not undergo changes, and the material point moves along the same path as in the case: $\mu_1, \mu_2, \eta_2 \neq 0$ and $\eta_1 = 0$. In both cases, the transformation coefficients B_{ij} are unchanged, since the sliding trajectories coincide.

If $\mu_1, \mu_2, \eta_2 \neq 0$ and $\eta_1 > 0$ (the positive additional friction), then the material point experiences greater resistance to motion in comparison with the case of negative additional friction $\eta_1 < 0$. It deals with $\eta_2 = \text{const}$ as well as with η_2 being the velocity function ($\eta_2 \neq \text{const}$).

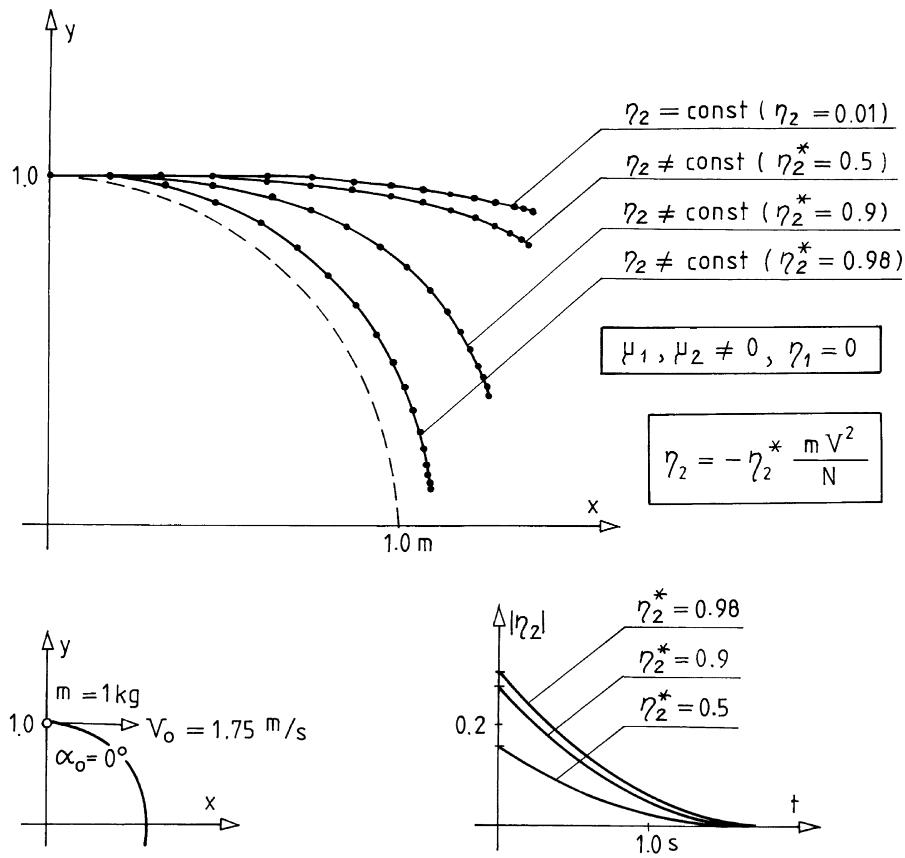


Fig. 4. Motion of a material point in a plane with friction properties which form concentric circles for different values of the constraints imposed on the motion in direction normal to the sliding path (η_2) and for $\eta_1 = 0$.

4. Conclusions

(1) The sliding path curvature induces: an additional resistance to sliding (η_1) and a reaction to the constraint normal to the sliding path (η_2). It produces an additional friction (η_1) and it can change essentially the sliding path curvature (η_2). These facts are illustrated by the sliding trajectories of the material point in the plane with non-homogeneous friction in the form of concentric circles.

(2) If the material point slides in the plane with non-homogeneous friction along the concentric circular trajectory excited by an initial velocity, then the constraint reaction normal to the sliding path is assumed to be equal and opposite to the centripetal force. In that case the coefficient η_2 is a function of the sliding velocity.

(3) The description of the material point sliding along the concentric circles is realized with the aid of the tensor **C** which defines anisotropic and non-homogeneous friction properties of the surface and the tensor **E** which defines constraints imposed on the sliding motion. The tensor

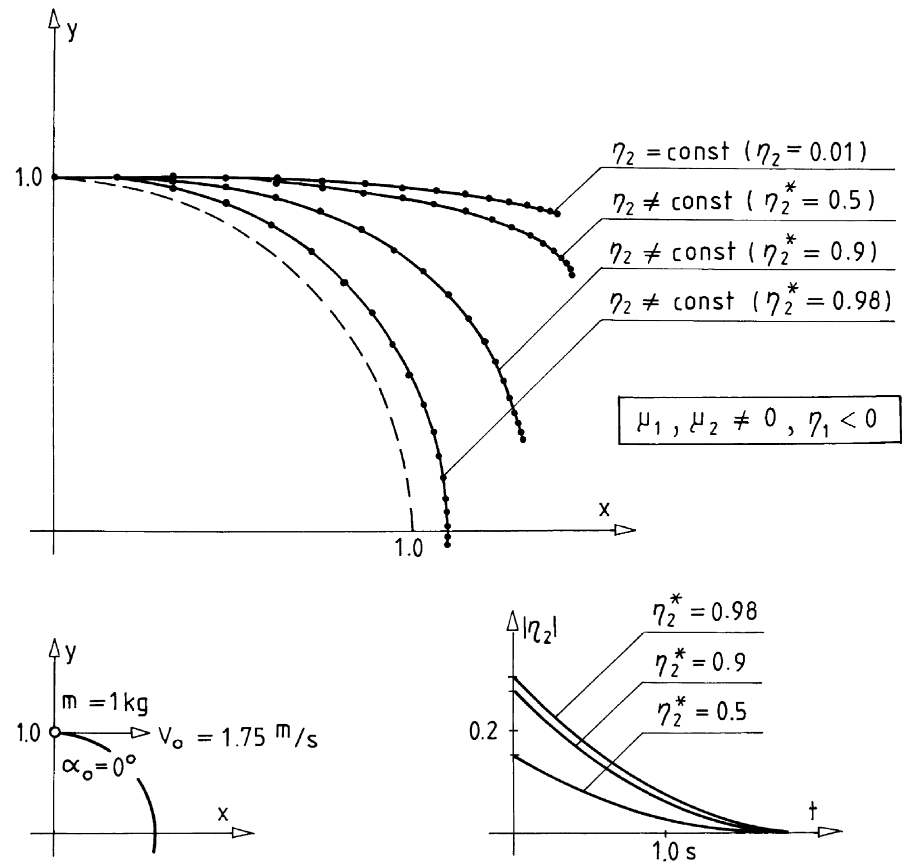


Fig. 5. Motion of a material point in a plane with friction properties which form concentric circles for different values of the constraints imposed on the motion in normal direction (η_2) and for the negative additional friction $\eta_1 = -0.01$ m.

E is an addition to the friction description with the aid of the tensor **C**, and its role vanishes in the case of rectilinear trajectories.

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